EGYPTIAN JOURNAL OF BASIC AND APPLIED SCIENCES 1 (2014) 71–76

Available online at www.sciencedirect.com

ScienceDirect

journal homepage: http://ees.elsevier.com/ejbas/default.asp

Two-dimensional time fractional-order biological population model and its analytical solution

Vineet K. Srivastava [a,*], *Sunil Kumar* [b], *Mukesh K. Awasthi* [c], *Brajesh Kumar Singh* [d]

[a] *ISRO Telemetry, Tracking and Command Network (ISTRAC), Bangalore 560058, Karnataka, India*
[b] *Department of Mathematics, National Institute of Technology, Jamshedpur 831014, Jharkhand, India*
[c] *Department of Mathematics, University of Petroleum and Energy Studies, Dehradun 247008, Uttarakhand, India*
[d] *Department of Mathematics, School of Allied Sciences, Graphic Era Hill University, Dehradun 248002, Uttarakhand, India*

ARTICLE INFO

Article history:
Received 22 November 2013
Received in revised form
6 March 2014
Accepted 8 March 2014
Available online 27 March 2014

Keywords:
GTFBPM
FRDTM
Mittag-leffler function
Caputo fractional derivative
Exact solution

ABSTRACT

In this article, a mathematical model has been developed for the generalized time fractional −order biological population model (GTFBPM). The fractional derivative has been described in the Caputo sense. The model has been solved by a recent approximate analytic method so called the fractional reduced differential transform method (FRDTM). Using this method, it is possible to find the exact solution as well as closed approximate solution of a differential equation. Three numerical examples of GTFBPM have been provided in order to check the effectiveness, accuracy and convergence of the method. The special advantage of using this computational technique is that it is very easy to implement and takes small size of computation contrary to other numerical methods while dealing complex and tedious physical problems arising in various branches of natural sciences and engineering.

1. Introduction

Various physical phenomena in natural sciences and engineering can be explained successfully by developing models using the fractional calculus theory. Fractional differential equations have achieved much more attention because fractional order system response ultimately converges to the integer order equations. The use of fractional differentiation for the mathematical modeling of real world physical problems such as the earthquake modeling, the traffic flow model with fractional derivatives, measurement of viscoelastic material properties, etc. has been widespread in the recent years. Before the nineteenth century, no analytical solution method was available for such type of equations even for the linear fractional differential

* *Corresponding author*. Tel./fax: +91 8050682145.
E-mail addresses: vineetsriiitm@gmail.com, vsrivastava107@gmail.com (V.K. Srivastava).
Peer review under responsibility of Mansoura University

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ejbas.2014.03.001
2314-808X/

equations. Recently, Keskin and Oturanc [1] developed the fractional reduced differential transform method (FRDTM) for the fractional differential equations and showed that FRDTM is the most easily implemented analytical method and gives the exact solution for both the linear and nonlinear differential equations. FRDTM is very reliable, efficient and effective powerful computational technique for solving physical problems; see references [2–5].

2. Generalized time fractional-order biological population model (GTFBPM)

Biological scientists believe that the dispersal or emigration plays a crucial role in the regulation of population of the species. The diffusion of a biological species in a region C is described by the three functions of position $\vec{x} = (x, y)$ in region C and time t [6] namely population density $p(\vec{x}, t)$, diffusion velocity $v(\vec{x}, t)$, and the population supply, $f(\vec{x}, t)$. The population density $p(\vec{x}, t)$ gives the number of individuals, per unit volume, at position $\vec{x}$ and time t; it's integral over any sub region D of region C gives the total population of D at time t. The entity $f(\vec{x}, t)$ gives the rate at which individuals are supplied, per unit volume, at position $\vec{x}$ by births and deaths. The diffusion velocity $v(\vec{x}, t)$ represents the average velocity of those individuals who occupy the position $\vec{x}$ at time t, and it describes the flow of population from point to point. The entities p, $\vec{v}$ and f must be consistent with the following law of population balance, for every regular sub region D of C and for all time t

$$\frac{d^\alpha}{dt^\alpha} \int_D p dV + \int_{\partial D} p \vec{v} \cdot \hat{n} dA = \int_D f dV, \tag{1}$$

where $\hat{n}$ is the outward unit normal to the boundary ∂D of D. The derivative in Eq. (1) has been taken in the Caputo derivative sense. From the Eq. (1), it is obvious that the rate of change of population of D plus the rate at which the individuals leave D across its boundary must be equal to the rate at which the individuals are supplied directly to D. By making the assumptions [8]

$$f = f(p), \quad \vec{v} = -\lambda(p)\nabla p, \tag{2}$$

where $\lambda(p) > 0$ for $p > 0$, and ∇ is the Laplace operator, the following two-dimensional nonlinear degenerate parabolic partial differential equation for the population density p can be obtained as

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial^2 \phi(p)}{\partial x^2} + \frac{\partial^2 \phi(p)}{\partial y^2} + f(p), \quad t \geq 0, \ x, y \in \mathbb{R}. \tag{3}$$

Eq. (3) is known as the time fractional-order biological population model (TFBPM). Gurney and Nisbet [7] employed $\phi(p)$, as a special case for the modeling of the population of animals. The movements are made generally either by mature animals driven by mature invaders or by young animals just reaching maturity moving out of their parental territory to establish breeding territory of their own. In both cases, it is much more plausible to assume that they will be directed towards nearby vacant territory. Therefore, in this model, movement takes place almost exclusively down the population density gradient and will be more rapid at high population densities than at low ones. To model this scenario, they considered a walk through a rectangular mesh, in which at each step an animal may either stay at its present location or may move in the direction of the lowest population density. The probability distribution among these two possibilities being determined by the magnitude of the population density gradient at the mesh side concerned. This model leads to Eq. (3) with $\phi(p)=p^2$, to the following equation

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial^2 p^2}{\partial x^2} + \frac{\partial^2 p^2}{\partial y^2} + f(p), \quad t \geq 0, \ x, y \in \mathbb{R}, \tag{4}$$

with the given initial condition $p(x,y,0)$. For $\alpha = 1$, Eq. (4) reduces to the normal biological population model (NBPM)

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p^2}{\partial x^2} + \frac{\partial^2 p^2}{\partial y^2} + f(p), \quad t \geq 0, \ x, y \in \mathbb{R}. \tag{5}$$

Some properties of Eq. (4) such as Holder estimates and its solutions have been studied in [8].

Three examples of constitutive equations for $f(p)$ can be given as

a) $f(p) = cp$, $c =$ constant, Malthusian Law [6].
b) $f(p) = c_1 p - c_2 p^2$, $c_1, c_2 =$ positive constants, Verhulst Law [8].
c) $f(p) = cp^\theta$, $(c > 0, 0 < \theta < 1)$, Porous media [9,10].

Let us consider a more general form of $f(p)$ as $f(p) = hp^a(1 - \lambda p^b)$, so that Eq. (4) becomes

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial^2 p^2}{\partial x^2} + \frac{\partial^2 p^2}{\partial y^2} + hp^a(1 - \lambda p^b), \quad t \geq 0, \ x, y \in \mathbb{R}, \tag{6}$$

where h, a, λ, b, are real numbers. If $h = c$, $a = 1$, $\lambda = 0$ and $h = c_1$, $a = b = 1$, $\lambda = c_2/c_1$, then Eq. (6) leads to Malthusian Law and Verhulst Law. The reason behind using the fractional-order differential equation is that it is naturally related to systems with memory which exists in most of the biological systems. Also they are closely related to fractals which are abundant in biological systems. The linear and nonlinear population systems were solved in [11,12] using variational iteration method (VIM), adomian decomposition method (ADM), homotopy analysis method (HAM), and homotopy perturbation method (HPM). The major disadvantage of these methods is that they require complex and large size of computations. To overcome from the complex computational efforts, FRDTM has been employed for solving the problem.

The purpose of this work is to solve the generalized time fractional–order biological population model (GTFBPM) using the fractional reduced differential transformation method. The obtained results are compared well with those obtained by VIM, ADM, HPM and HAM.

3. Fractional calculus theory

In this section, we present some notations, definitions and preliminary facts that will be used further in this study. Fractional calculus theory is more than 200 years old theory present in the literature. Several definitions of fractional integrals and derivatives have been proposed but the first major contribution to give proper and most meaningful definition is due to Liouville as follows.

Definition 3.1. A real function $f(x)$, $x > 0$ is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number q $(>\mu)$, such that

$f(x) = x^q g(x)$, where $g(x) \in C[0,\infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 3.2. For a function f, the Riemann-Liouville fractional integeral operator [15] of order $\alpha \geq 0$, is defined as

$$\begin{cases} J^\alpha f(x) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0,\ x > 0, \\ J^0 f(x) = f(x). \end{cases} \tag{7}$$

The Riemann-Liouville derivative has major drawbacks while modeling the real world problems with fractional differential equations. To overcome this discrepancy, Caputo and Mainardi [16] proposed a modified fractional differentiation operator D^α in his work on the theory of viscoelasticity. The Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations.

Definition 3.3. The fractional derivative of $f(x)$ in the Caputo sense [17] is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{8}$$

for $m - 1 < \alpha \leq m$, $m \in \mathbb{N}, x > 0$, $f \in C_{-1}^m$.

The fundamental properties of the Caputo fractional derivative are given in the following lemma.

Lemma. If $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$ and $f \in C_\mu^m$, $\mu \geq -1$, then

$$\begin{cases} D^\alpha J^\alpha f(x) = f(x),\ x > 0, \\ J^\alpha D^\alpha f(x) = f(x) - \sum\limits_{k=0}^{m} f^{(k)}(0^+)\dfrac{x^k}{k!},\ x > 0. \end{cases} \tag{9}$$

In this study, the Caputo fractional derivative has been chosen because it allows traditional initial and boundary conditions to be included in the formulation of the physical problems. Some other important characteristics of fractional derivatives can be seen in [17,18].

4. Fractional reduced differential transform method (FRDTM)

In this section, the basic definitions and properties of the fractional reduced differential transform method have been described.

Consider a function of two variables $w(x,t)$, and assume that it can be represented as a product of two single-variable functions, i.e. $w(x,t) = F(x)G(t)$. On the basis of the properties of the one-dimensional differential transform, the function $w(x,t)$ can be represented as

$$w(x,t) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} W(i,j)x^i t^j, \tag{10}$$

where $W(i,j) = F(i)G(j)$ is called the spectrum of $w(x,t)$.

Let R_D denotes the reduced differential transform operator and R_D^{-1} the inverse reduced differential transform operator [4]. The basic definitions and operations of the reduced differential transform are introduced as follows.

Definition 4.1. If $w(x,t)$ is analytic and continuously differentiable with respect to space variable x and time variable t in the domain of interest, then the t-dimensional spectrum function

$$W_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^k}{\partial t^k} w(x,t)\right]_{t=t_0} \tag{11}$$

is the fractional reduced transformed function of $w(x,t)$, where α is a parameter which describes the order of time-fractional derivative. In this paper, (lowercase) $w(x,t)$ represents the original function while (uppercase) $W_k(x)$ stands for the fractional reduced transformed function. The differential inverse transform of $W_k(x)$ is defined as

$$w(x,t) = \sum_{k=0}^{\infty} W_k(x)(t-t_0)^{k\alpha}. \tag{12}$$

combining Eqs. (11) and (12), it can be found that

$$w(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^k}{\partial t^k} w(x,t)\right]_{t=t_0} (t-t_0)^{k\alpha}. \tag{13}$$

when $t = 0$, Eq. (13) reduces to

$$w(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^k}{\partial t^k} w(x,t)\right]_{t=0} t^{k\alpha}. \tag{14}$$

From the above definition, it can be found that the concept of the fractional reduced differential transform is derived from the power series expansion of a function.

Definition 4.2. If $u(x,t) = R_D^{-1}[U_k(x)]$, $v(x,t) = R_D^{-1}[V_k(x)]$ and the convolution $\otimes$ denotes the fractional reduced differential transform version of the multiplication, then the fundamental operations of the fractional reduced differential transform have been expressed in Table 1.

In Table 1 Γ, represents the Gama function, which is defined as

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt, z \in \mathbb{C}. \tag{15}$$

here we notice that the Gama function is the continuous extension to the factorial function [3].

The recursive relationship $\Gamma(z+1) = z\Gamma(z), z > 0$ can be used to compute the value of the gamma function of all real numbers (except the non-positive integers) by knowing only the value of the gamma function between 1 and 2.

Table 1 – Fundamental operations of the fractional reduced differential transform method.

Original function	Fractional reduced differential transformed function
$R_D\ [u(x,t)v(x,t)]$	$U_k(x) \otimes V_k(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$R_D\ [\alpha u(x,t) \pm \beta v(x,t)]$	$\alpha U_k(x) \pm \beta V_k(x)$
$R_D\ [(\partial^l/\partial x^l)\ u(x,t)]$	$[(k+l)!/k!]\ (\partial^l/\partial x^l)\ U_{k+l}(x)$
$R_D\ [(\partial^{N\alpha}/\partial t^{N\alpha})\ u(x,t)]$	$[(\Gamma(k\alpha + N\alpha + 1))/(\Gamma(k\alpha+1))]\ U_{k+N}(x)$
$R_D\ [x^m t^n u(x,t)]$	$x^m U_{k-n}(x)$
$R_D\ [e^{\lambda t}]$	$\lambda^k/k!$
$R_D\ [\sin\ (wt+\alpha)]$	$(w^k/k!)\sin[(\pi k/2!) + \alpha]$
$R_D\ [\cos\ (wt+\alpha)]$	$(w^k/k!)\cos[(\pi k/2!) + \alpha]$

5. Numerical examples

In this section, we describe the method explained in the Section 4 by taking three numerical examples to validate the efficiency and reliability of FRDTM for the GTFBPM.

Example 5.1. Consider the following linear time fractional-order biological population model

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial^2 p^2}{\partial x^2} + \frac{\partial^2 p^2}{\partial y^2} + hp, \tag{16}$$

subject to the initial condition

$$p(x,y,0) = \sqrt{xy}. \tag{17}$$

Applying the FRDTM to Eq. (16), we obtain the following recurrence relation

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)} P_{k+1}(x,y) = \frac{\partial^2}{\partial x^2}\left[\sum_{r=0}^{k} P_r(x,y)P_{k-r}(x,y)\right] + \frac{\partial^2}{\partial y^2}\left[\sum_{r=0}^{k} P_r(x,y)P_{k-r}(x,y)\right] + hP_k(x,y). \tag{18}$$

Using the FRDTM to the initial condition (17), we get

$$P_0(x,y) = \sqrt{xy}. \tag{19}$$

using Eq. (19) in Eq. (18), we get the following $P_k(x,y)$ values successively

$$P_1(x,y) = \frac{h}{\Gamma(1+\alpha)}\sqrt{xy};\ P_2(x,y) = \frac{h^2}{\Gamma(1+2\alpha)}\sqrt{xy};\ P_3(x,y) = \frac{h^3}{\Gamma(1+3\alpha)}\sqrt{xy};$$
$$P_4(x,y) = \frac{h^4}{\Gamma(1+4\alpha)}\sqrt{xy};\ldots;P_k(x,y) = \frac{h^k}{\Gamma(1+k\alpha)}\sqrt{xy}. \tag{20}$$

Using the differential inverse transform of $P_k(x,y)$, $k = 1,2,3,\ldots$, we get

$$\begin{aligned} p(x,y,t) &= \sum_{k=0}^{\infty} P_k(x,y)t^{k\alpha} = P_0(x,y) + P_1(x,y)t^\alpha + P_2(x,y)t^{2\alpha} + P_3(x,y)t^{3\alpha} + \ldots \\ &= \sqrt{xy}\left(1 + \frac{h}{\Gamma(1+\alpha)}t^\alpha + \frac{h^2}{\Gamma(1+2\alpha)}t^{2\alpha} + \ldots + \frac{h^k}{\Gamma(1+k\alpha)}t^{3\alpha} + \ldots\right) \\ &= \sqrt{xy}\left[\sum_{k=0}^{\infty}\frac{(ht^\alpha)^k}{\Gamma(1+k\alpha)}\right]. \end{aligned} \tag{21}$$

The exact solution can be expressed as

$$p(x,y,t) = \sqrt{xy}\,E_\alpha(ht^\alpha), \tag{22}$$

where $E_\alpha(ht^\alpha)$ is Mittag-leffler function, defined as $E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)}$, which is a advanced form of *exp* (z), and reduces to *exp* (z) when $\alpha \to 1$.

The same result was obtained by Y. Liu et al. [13] using HPM and Arafa et al. [14] using HAM. When $\alpha \to 1$ in Eq. (22), we get

$$p(x,y,t) = \sqrt{xy}\sum_{k=0}^{\infty}\frac{(ht)^k}{\Gamma(1+k)} = (\sqrt{xy})e^{ht}, \tag{23}$$

as the exact solution for the standard form of the biological population equation (SBPE), same result was obtained for this standard form by Roul [12] using HPM and Shakeri et at [11]. using VIM and ADM with parameter $h = 1/5$.

Example 5.2. Consider the following linear time fractional-order biological population model

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial^2 p^2}{\partial x^2} + \frac{\partial^2 p^2}{\partial y^2} + p, \tag{24}$$

with the initial condition

$$p(x,y,0) = \sqrt{\sin x \sinh y}. \tag{25}$$

Applying the FRDTM to Eq. (24), we obtain the following recurrence equation:

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)} P_{k+1}(x,y) = \frac{\partial^2}{\partial x^2}\left[\sum_{r=0}^{k} P_r(x,y)P_{k-r}(x,y)\right] + \frac{\partial^2}{\partial y^2}\left[\sum_{r=0}^{k} P_r(x,y)P_{k-r}(x,y)\right] + P_k(x,y). \tag{26}$$

using the FRDTM to the initial condition (25), we get

$$P_0(x,y) = \sqrt{\sin x \sinh y}. \tag{27}$$

using Eq. (27) in Eq. (26), we get the following $P_k(x,y)$ values successively

$$\begin{aligned}&P_1(x,y)=\frac{1}{\Gamma(1+\alpha)}\sqrt{\sin x\sinh y};\ \ P_2(x,y)=\frac{1}{\Gamma(1+2\alpha)}\sqrt{\sin x\sinh y};\\&P_3(x,y)=\frac{1}{\Gamma(1+3\alpha)}\sqrt{\sin x\sinh y};\ \ P_4(x,y)=\frac{1}{\Gamma(1+4\alpha)}\sqrt{\sin x\sinh y};\\&P_k(x,y)=\frac{1}{\Gamma(1+k\alpha)}\sqrt{\sin x\sinh y}.\end{aligned}\tag{28}$$

using the differential inverse reduced transform of $P_k(x,y)$, $k=1,2,3,\ldots$, we have

$$\begin{aligned}p(x,y,t)&=\sum_{k=0}^{\infty}P_k(x,y)t^{k\alpha}=P_0(x,y)+P_1(x,y)t^{\alpha}+P_2(x,y)t^{2\alpha}+P_3(x,y)t^{3\alpha}+\ldots\\&=\sqrt{\sin x\sinh y}\left(1+\frac{1}{\Gamma(1+\alpha)}t^{\alpha}+\frac{1}{\Gamma(1+2\alpha)}t^{2\alpha}+\ldots+\frac{1}{\Gamma(1+k\alpha)}t^{k\alpha}+\ldots\right)\\&=\sqrt{\sin x\sinh y}\left[\sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(1+k\alpha)}\right].\end{aligned}\tag{29}$$

thus, the exact solution can be given as

$$p(x,y,t)=\sqrt{\sin x\sinh y}\,E_{\alpha}(t^{\alpha}).\tag{30}$$

the same solution was obtained by Arafa et al. [14] using HAM. When $\alpha\to 1$ in Eq. (30), we get

$$p(x,y,t)=\left(\sqrt{\sin x\sinh y}\right)e^{t}.\tag{31}$$

The same exact solution had been obtained by Roul [12] using HPM for the standard biological population model (SBPM).

Example 5.3. Consider the following generalized nonlinear time fractional-order biological population model

$$\frac{\partial^{\alpha}p}{\partial t^{\alpha}}=\frac{\partial^2p^2}{\partial x^2}+\frac{\partial^2p^2}{\partial y^2}+p(1-\lambda p),\tag{32}$$

subject to the initial condition

$$p(x,y,0)=\exp\left[\frac{1}{2}\sqrt{\frac{\lambda}{2}}(x+y)\right].\tag{33}$$

Applying the FRDTM to Eq. (32), we obtain the following iteration formula

$$\begin{aligned}\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}P_{k+1}(x,y)=&\frac{\partial^2}{\partial x^2}\left[\sum_{r=0}^{k}P_r(x,y)P_{k-r}(x,y)\right]\\&+\frac{\partial^2}{\partial y^2}\left[\sum_{r=0}^{k}P_r(x,y)P_{k-r}(x,y)\right]\\&-P_k(x,y)-\lambda\sum_{r=0}^{k}P_r(x,y)P_{k-r}(x,y).\end{aligned}\tag{34}$$

using the FRDTM to the initial condition (33), we get

$$P_0(x,y)=\exp\left[\frac{1}{2}\sqrt{\frac{\lambda}{2}}(x+y)\right].\tag{35}$$

using Eq. (35) in Eq. (34), the following $P_k(x,y)$ values are obtained successively

$$\begin{aligned}&P_1(x,y)=\frac{1}{\Gamma(1+\alpha)}\exp\left[\frac{1}{2}\sqrt{\frac{\lambda}{2}}(x+y)\right];\ \ P_2(x,y)=\frac{1}{\Gamma(1+2\alpha)}\exp\left[\frac{1}{2}\sqrt{\frac{\lambda}{2}}(x+y)\right];\\&P_3(x,y)=\frac{1}{\Gamma(1+3\alpha)}\exp\left[\frac{1}{2}\sqrt{\frac{\lambda}{2}}(x+y)\right];\ P_4(x,y)=\frac{1}{\Gamma(1+4\alpha)}\exp\left[\frac{1}{2}\sqrt{\frac{\lambda}{2}}(x+y)\right];\\&P_k(x,y)=\frac{1}{\Gamma(1+k\alpha)}\exp\left[\frac{1}{2}\sqrt{\frac{\lambda}{2}}(x+y)\right].\end{aligned}\tag{36}$$

Using the differential inverse reduced transform of $P_k(x,y)$, $k = 1,2,3,\ldots$, we get

$$
\begin{aligned}
p(x,y,t) &= \sum_{k=0}^{\infty} P_k(x,y)t^{k\alpha} = P_0(x,y) + P_1(x,y)t^{\alpha} + P_2(x,y)t^{2\alpha} + P_3(x,y)t^{3\alpha} + \ldots \\
&= \exp\left[\frac{1}{2}\sqrt{\frac{\lambda}{2}}(x+y)\right]\left(1 + \frac{1}{\Gamma(1+\alpha)}t^{\alpha} + \frac{1}{\Gamma(1+2\alpha)}t^{2\alpha} + \ldots + \frac{1}{\Gamma(1+k\alpha)}t^{k\alpha} + \ldots\right) \\
&= \exp\left[\frac{1}{2}\sqrt{\frac{\lambda}{2}}(x+y)\right]\left[\sum_{k=0}^{\infty}\frac{(t^{\alpha})^k}{\Gamma(1+k\alpha)}\right].
\end{aligned}
\tag{37}
$$

Hence the closed form solution is given by

$$p(x,y,t) = \exp\left[\frac{1}{2}\sqrt{\frac{\lambda}{2}}(x+y)\right]E_{\alpha}(t^{\alpha}). \tag{38}$$

The same result was obtained by Arafa et al. [14] using HAM. As $\alpha \to 1$ in Eq. (38), we have

$$p(x,y,t) = \exp\left(\left[\frac{1}{2}\sqrt{\frac{\lambda}{2}}(x+y)\right] + t\right). \tag{39}$$

which is the same exact solution obtained by Roul [12] using HPM, and also by Shakeri et at [11]. using VIM and ADM.

6. Conclusions

In this work, the fractional reduced differential transform method has been implemented for a degenerate Caputo time-fractional order parabolic partial differential equation arising in the spatial diffusion biological populations. The solution obtained by the FRDTM is an infinite power series for appropriate initial condition which finds the solution without any discretization, transformation, perturbation, or restrictive conditions. We have also illustrated three numerical examples considering the situations of both linear as well nonlinear phenomenon of GTFBPM to study the effectiveness and accurateness of the method. The solutions obtained by the FRDTM are in excellent agreement with those obtained by the HPM, HAM, VIM and ADM. However, computations show that the FRDTM is very easy to implement and needs small size of computation contrary to HPM, HAM, VIM and ADM.

REFERENCES

[1] Keskin Y, Oturanc G. Reduced differential transform method: a new approach to factional partial differential equations. Nonlinear Sci Lett A 2010;1:61–72.

[2] Gupta PK. Approximate analytical solutions of fractional Benney-Lin equation by reduced differential transform method and the homotopy perturbation method. Comp Math Appl 2011;58:2829–42.

[3] Srivastava VK, Awasthi MK, Tamsir M. RDTM solution of Caputo time fractional-order hyperbolic telegraph equation. AIP Adv 2013;3:032142.

[4] Srivastava VK, Awasthi MK, Chaurasia RK, Tamsir M. The telegraph equation and its solution by reduced differential transform method. Model Simul Eng 2013;2013. Article ID 746351.

[5] Srivastava VK, Awasthi MK, Kumar S. Analytical approximations of two and three dimensional time-fractional telegraphic equation by reduced differential transform method. Egypt J Basic Appl Sci; 2014. http://dx.doi.org/10.1016/j.ejbas.2014.01.002.

[6] Gurtin ME, Maccamy RC. On the diffusion of biological population. Math Biol Sci 1977;54(33):35–49.

[7] Gurney WSC, Nisbet RM. The regulation of inhomogenous populations. J Theor Biol 1975;52:441–57.

[8] Lu YG. Hölder estimate of solutions of biological population equations. Appl Math Lett 2000;13:123–6.

[9] Bear J. Dynamics of fluids in porous media. New York: American Elsevier; 1972.

[10] Okubo A. Diffusion and ecological problem, mathematical models. Biomathematics 10. Berlin: Springer; 1980.

[11] Shakeri F, Dehghan M. Numerical solution of a biological population model using He's variational iteration method. Comput Math Appl 2006;54:1197–209.

[12] Roul P. Application of homotopy perturbation method to biological population model. Appl Appl Math 2010;10:1369–78.

[13] Liu Y, Li Z, Zhang Y. Homotopy perturbation method to fractional biological population equation. Fract Diff Calc 2011;1:117–24.

[14] Arafa AA, Rida SZ, Mohamed H. Homotopy analysis method for solving biological population model. Commun Theor Phys 2011;56:797–800.

[15] Millar KS, Ross B. An introduction to the fractional calculus and fractional differential equations. New York: Wiley; 1993.

[16] Caputo M, Mainardi F. Linear models of dissipation in anelastic solids. Rivista del Nuovo Cimento 1971;1:161–98.

[17] Podlubny I. Fractional differential equations. San Diego: Academic Press; 1999.

[18] Hilfer R. Applications of fractional calculus in physics. Singapore: World Scientific; 2000.